

# The duality morphism

$G$  connected reductive gp/ $k$ ,

$\uparrow$   $\text{char}(k) \neq |W|$

$T, \alpha, \Phi, \dots$

$\text{Higgs}_G^{\vee} / \mathcal{B}^{\circ}$

$\downarrow$

$\mathcal{B}^{\circ} = \mathcal{B} \setminus \Delta$

torsor under  $\mathcal{P} := \mathcal{P}_G / \mathcal{B}^{\circ}$

$\check{G}$  Langlands dual

$\uparrow$

$\check{T}, \check{\alpha}, \check{\Phi}, \dots$

$\text{Higgs}_{\check{G}}^{\vee} / \check{\mathcal{B}}^{\circ}$

$\downarrow$

$\check{\mathcal{B}}^{\circ} = \check{\mathcal{B}} \setminus \check{\Delta}$

torsor under  $\check{\mathcal{P}} := \check{\mathcal{P}}_{\check{G}} / \check{\mathcal{B}}^{\circ}$

Main goal:

Thm (Donagi-Pantev, Chen-Zhu)  
 $\exists$  iso of Picard stacks.

$$\begin{array}{ccc}
 \mathcal{P}^{\vee} := \text{Hom}_{\text{PS}}(\mathcal{P}, \mathcal{B}G_m) & \xrightarrow{\sim} & \check{\mathcal{P}} \\
 \downarrow & & \downarrow \\
 \mathcal{B}^{\circ} & \xrightarrow{\sim} & \check{\mathcal{B}}^{\circ}
 \end{array}$$

Today: Definition of  $\mathcal{D}$  & duality for  $\pi_0(\dots)$ .

Ex.  $G = \check{G} = GL_n$

$\gamma_b \rightarrow X$  spectral curve over  $b \in B^\circ(k) = \check{B}^\circ(k)$

$\Rightarrow \mathcal{P}_b^\vee \simeq \text{Pic}(\gamma_b)^\vee \xrightarrow{\sim} \text{Pic}(\gamma_b) \simeq \check{\mathcal{P}}_b$

autoduality induced by Abel-Jacobi map

$AJ: \gamma_b \rightarrow \text{Pic}(\gamma_b), p \mapsto \mathcal{O}_{\gamma_b}(p).$

In general:

0. Duality for cameral covers
1. Abel-Jacobi map
2. Definition of  $\mathcal{D}$
3. Reduction to coarse moduli
4. Next time: Coarse moduli

0. Duality for cameral covers

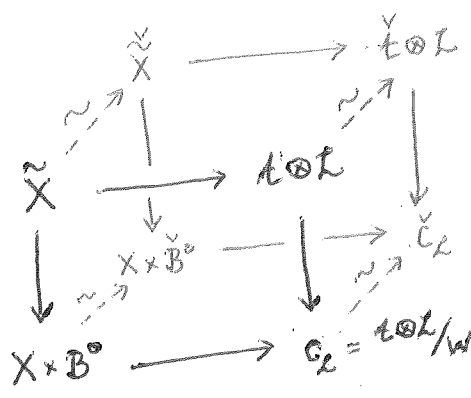
Lemma 1.  $\exists$  iso of universal cameral curves

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\sim} & \check{X} \\ \downarrow & & \downarrow \\ B^\circ & \xrightarrow{\sim} & \check{B}^\circ \end{array}$$

Proof. Choose invariant scalar product on  $\mathfrak{g}$

to get  $W$ -equivariant iso  $\mathfrak{k} \xrightarrow{\sim} \check{\mathfrak{k}}$  preserving the discriminant.

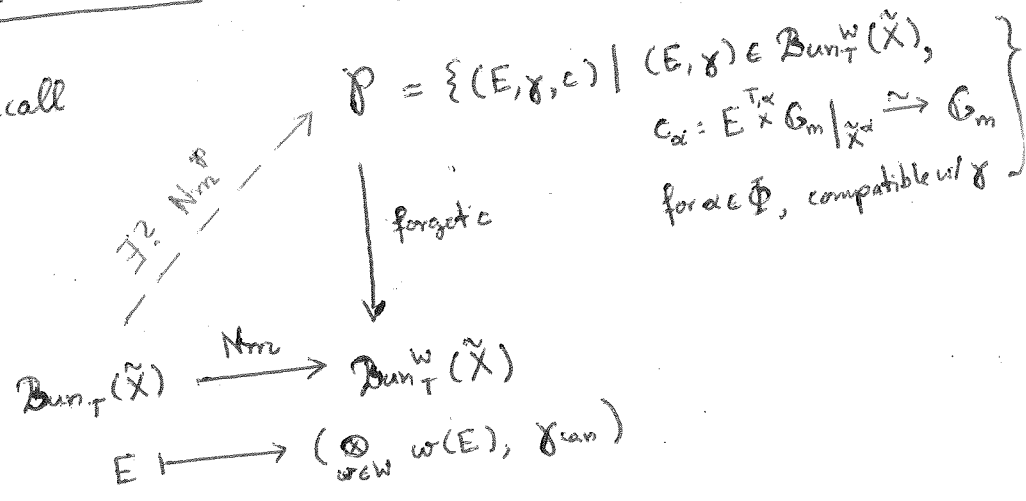
Then



□

### 1. The Abel-Jacobi map

Recall



Lemma 2.  $\exists$  natural lift  $\text{Nm}^p$ .

Proof.

$$\bigotimes_{\omega \in W} (\omega(E) \times_{T_\alpha} \mathbb{G}_m) \big|_{\tilde{X}^\alpha} \simeq \bigotimes_{\omega \in \langle S_\alpha \rangle} \underbrace{((\omega(E) \otimes_{S_\alpha} \omega(E)) \times_{T_\alpha} \mathbb{G}_m) \big|_{\tilde{X}^\alpha}}_{= \mathbb{G}_m, \text{ canonically}}$$

③

□

Put  $\Lambda := X_*(T) = \text{Hom}(G_m, T)$

and define  $AJ$  by

$$(x, \lambda) \mapsto \mathcal{O}_{\tilde{X}}(\lambda x) := \mathcal{O}_{\tilde{X}}(x) \otimes_{G_m, \lambda} T$$

$$\begin{array}{ccc} \tilde{X} \times \Lambda & \longrightarrow & \text{Bun}_T(\tilde{X}) \\ & \searrow \text{AJ} & \downarrow \text{Nm}^P \\ & & P \end{array}$$

"Abel-Jacobi map"

Rem. 3

For  $\alpha \in \Phi$

$x \in \tilde{X}^\alpha$

one has  $AJ(x, \check{\alpha}) = e$

Since  $\bigotimes_{w \in W} w(\mathcal{O}(\check{\alpha}x)) \simeq \bigotimes_{w \in W/\langle s_\alpha \rangle} w(\mathcal{O}(\check{\alpha}x) \otimes \mathcal{O}(s_\alpha(\check{\alpha})x))$   
 trivial.

Lemma 4

$AJ$  is multiplicative &  $W$ -equivariant,

ie we have natural trafs  $\mu$  and  $\gamma$

as follows =

$$\begin{array}{ccc} \tilde{X} \times \Lambda \times \Lambda & \xrightarrow{(AJ, AJ)} & P \times P \\ \downarrow & \swarrow \mu & \downarrow \\ \tilde{X} \times \Lambda & \xrightarrow{AJ} & P \end{array} \quad \left| \quad \begin{array}{ccc} \tilde{X} \times \Lambda \times W & \xrightarrow{ps_{12}} & \tilde{X} \times \Lambda \\ \downarrow \text{act} & \searrow \gamma & \downarrow AJ \\ \tilde{X} \times \Lambda & \xrightarrow{AJ} & P \end{array}$$

□

## 2. Definition of $\mathcal{D}: \mathcal{P}^v \rightarrow \check{\mathcal{P}}$

Prop. 5  $\exists$  natural morphism  $\mathcal{P}^v \xrightarrow{\mathcal{D}} \check{\mathcal{P}}$   
over  $\mathcal{B}^0 \xrightarrow{\sim} \check{\mathcal{B}}^0$ .

Proof.

• Recall

$$\mathcal{P}^v := \text{Hom}_{\mathcal{P}\mathcal{S}}(\mathcal{P}, \mathcal{B}\mathcal{G}_m) \simeq \text{Pic}^m(\mathcal{P})$$

where  $\text{Pic}^m(\mathcal{P})(S) := \{(\mathcal{M}, \mu) \mid \mathcal{M} \in \text{Pic}(\mathcal{P})(S),$   
 $\mu: \mathcal{M} \boxtimes \mathcal{M} \xrightarrow{\sim} \mathcal{O}^* \mathcal{M}\}$   
"multiplicative line bundles" w/ natural compatibilities

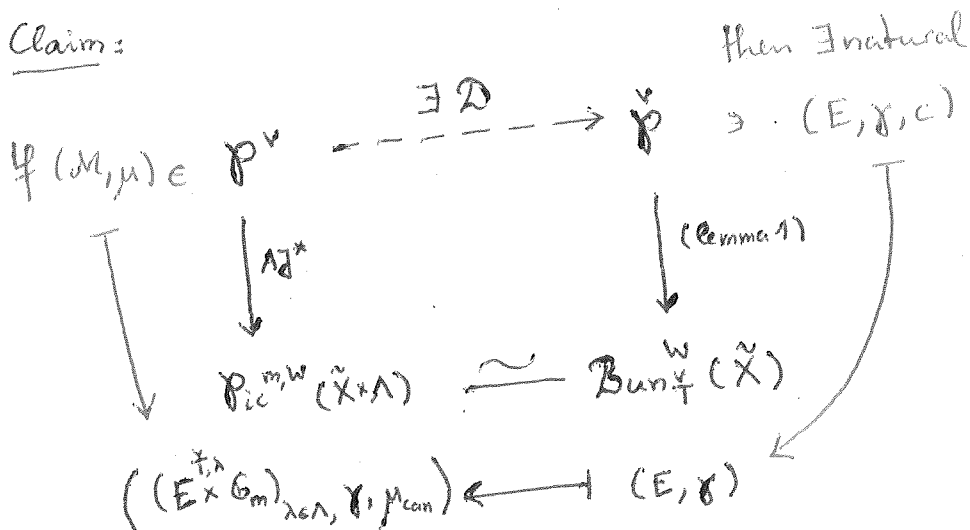
• Similarly, put

$$\text{Pic}^{m,W}(\check{X} \times \Lambda)(S) := \{(\mathcal{M}, \gamma, \mu) \mid (\mathcal{M}, \gamma) \in \text{Pic}^W(\check{X} \times \Lambda)(S),$$
  
 $\mu: \mathcal{M} \boxtimes \mathcal{M} \xrightarrow{\sim} \mathcal{O}^* \mathcal{M}$   
compatible with  $\gamma\}$

• Lemma 4 gives:

$$\mathcal{A}\mathcal{D}^*: \mathcal{P}^v \longrightarrow \text{Pic}^{m,W}(\check{X} \times \Lambda).$$

Claim:



Indeed:

$(M, \mu)$  multiplicative &  $A\mathcal{D}(x, \check{\alpha}) = e \quad \forall x \in \check{X}^{\check{\alpha}}$   
 $\alpha \in \Phi$   
 (remark 3)

$\Rightarrow \exists$  natural iso's  $\mathcal{M}_{A\mathcal{D}(x, \check{\alpha})} \xrightarrow{\sim} G_m$

$\Rightarrow \exists$  natural iso's

$$c_{\check{\alpha}} = E^{\check{\gamma}, \check{\alpha}} \times G_m \Big|_{\check{X}^{\check{\alpha}}} \xrightarrow{\sim} G_m$$

& we can take  $c = (c_{\check{\alpha}})_{\check{\alpha} \in \Phi}$ .



### 3. Reduction to coarse moduli

Basic Fact:  $\mathcal{P}$  is a Beilinson 1-motive /  $B^0$ ,  
more precisely:

$$\begin{array}{ccccc}
 W_0 & \subset & W_1 & \subset & W_2 = \mathcal{P} \\
 \parallel & & \downarrow & \downarrow & \\
 \mathcal{BZ}(G) & & \text{an abelian scheme / } B^0 & \text{gr}_2^W = \pi_0(\mathcal{P}) & \\
 & & & \vdots & \\
 & & & \Lambda / \mathbb{Z}\Phi & =: \pi_1(G)
 \end{array}$$

Idea of proof.

↖ coarse moduli space

know  $\text{ker}(\mathcal{P} \rightarrow \mathcal{P}) = \mathcal{BZ}(G)$ ,

so main point is to see  $\pi_0(\mathcal{P}) = \pi_0(\mathcal{P}_b) = \pi_1(G)$   
 $\forall b \in B^0(k)$ .

For this use

$$\Lambda = \pi_0(\tilde{X}_b \times \Lambda) \xrightarrow{\pi_0(\text{Adj})} \pi_0(\mathcal{P}_b)$$

$$\begin{array}{c}
 \downarrow \\
 \Downarrow \\
 \pi_1(G) = \Lambda / \mathbb{Z}\Phi
 \end{array}$$

↗  $\exists \varphi$

since  $\text{Adj}(x, \check{x}) = 0$   
 $\forall x \in \tilde{X}^d, a \in \Phi$   
(remark 3).

See Faltings,  
J. Alg. Geom. 2 (1993)



Cor. 6

$\mathcal{D}$  induces iso's

$$W_0(\mathcal{P}^\vee) \xrightarrow{\sim} W_0(\check{\mathcal{P}})$$

$$\pi_0(\mathcal{P}^\vee) \xrightarrow{\sim} \pi_0(\check{\mathcal{P}}).$$

Proof.

• Duality  $\Rightarrow$  enough to discuss  $\pi_0(\dots)$ .

• As abstract groups,

$$\pi_0(\mathcal{P}^\vee) \cong \text{Hom}(\underbrace{\text{Aut}_{\mathcal{P}}(e)}_{\cong Z(G)}, G_m)$$

$$\cong \pi_1(\check{G})$$

$$\cong \pi_0(\check{\mathcal{P}})$$

$\Rightarrow$  enough to show  $\pi_0(\mathcal{D})$  is epi,

since both sides are fin. gen. abelian gps.

• Epi follows from

$$\check{\Lambda} = \text{Hom}(\tau, G_m)$$

$$\begin{array}{ccc} \check{X} \times \check{\Lambda} & \xrightarrow{\psi} & \check{\mathcal{P}} \\ \exists \psi \swarrow & \searrow \text{epi on } \pi_0(\dots) & \\ \mathcal{P}^\vee & \xrightarrow{\mathcal{D}} & \check{\mathcal{P}} \end{array}$$

with  $\psi$  constructed from universal line bundle on  $(\check{X} \times \check{\Lambda}) \times \text{Bun}_\tau(\check{X})$

