05/07/16

The duality morphism

G connected reductive gp$/k$, char$(k) \nmid \lvert W \rvert$ $\begin{array}{c} \uparrow \\ T,\ \mathfrak{t},\ \Phi,\ \ldots \end{array}$ $\mathcal{H}iggs_G\lvert_{\mathcal{B}^\circ}$ $\downarrow$ $\mathcal{B}^\circ = \mathcal{B} \setminus \Delta$ torsor under $\mathcal{P} := \mathcal{P}_G\lvert_{\mathcal{B}^\circ}$	$\check{G}$ Langlands dual $\begin{array}{c} \uparrow \\ \check{T},\ \check{\mathfrak{t}},\ \Phi^\vee,\ \ldots \end{array}$ $\mathcal{H}iggs_{\check{G}}\lvert_{\check{\mathcal{B}}^\circ}$ $\downarrow$ $\check{\mathcal{B}}^\circ = \check{\mathcal{B}} \setminus \check{\Delta}$ torsor under $\check{\mathcal{P}} := \mathcal{P}_{\check{G}}\lvert_{\check{\mathcal{B}}^\circ}$

Main goal:

<u>Thm</u> (Donagi-Pantev, Chen-Zhu)
$\exists$ iso of Picard stacks

$$\begin{array}{ccc} \mathcal{P}^\vee := \mathcal{H}om_{ps}(\mathcal{P}, B\mathbb{G}_m) & \xrightarrow[\mathcal{D}]{\sim} & \check{\mathcal{P}} \\ \downarrow & & \downarrow \\ \mathcal{B}^\circ & \xrightarrow{\sim} & \check{\mathcal{B}}^\circ \end{array}$$

Today: Definition of $\mathcal{D}$ & duality for $\pi_0(\ldots)$.

Ex. $G = \check{G} = Gl_n$

$Y_b \longrightarrow X$ spectral curve over $b \in B^0(k) = \check{B}^0(k)$

$$\Rightarrow \mathcal{P}_b^\vee \cong \mathrm{Pic}(Y_b)^\vee \xrightarrow{\sim} \mathrm{Pic}(Y_b) \cong \check{\mathcal{P}}_b$$

autoduality induced by Abel-Jacobi map

$$AJ : Y_b \longrightarrow \mathrm{Pic}(Y_b), \quad p \mapsto \mathcal{O}_{Y_b}(p).$$

In general:

0. Duality for cameral covers
1. Abel-Jacobi map
2. Definition of $\mathcal{D}$
3. Reduction to coarse moduli
4. Next time: Coarse moduli

0. Duality for cameral covers

Lemma 1. $\exists$ iso of universal cameral curves

$$\begin{tikzcd}
\tilde{X} \arrow[r, "\sim"] \arrow[d] & \check{\tilde{X}} \arrow[d] \\
B^0 \arrow[r, "\sim"] & \check{B}^0
\end{tikzcd}$$

Proof. Choose invariant scalar product on $\mathfrak{g}$ to get W-equivariant iso $\mathfrak{t} \xrightarrow{\sim} \check{\mathfrak{t}}$ preserving the discriminant.

Then

$$\begin{tikzcd}[row sep=small, column sep=small]
 & \check{\tilde{X}} \arrow[rr] \arrow[dd] & & \check{t}\otimes L \arrow[dd] \\
\tilde{X} \arrow[rr] \arrow[dd] \arrow[ur, dashed, "\sim"] & & t\otimes L \arrow[ur, dashed, "\sim"] \arrow[dd] & \\
 & X\times \check{B}^\circ \arrow[rr] & & \check{C}_L \\
X\times B^\circ \arrow[rr] \arrow[ur, dashed, "\sim"] & & C_L = t\otimes L/W \arrow[ur, dashed, "\sim"] &
\end{tikzcd}$$

$\square$

1. The Abel-Jacobi map

Recall

$$\mathcal{P} = \left\{ (E,\gamma,c) \;\middle|\; \begin{array}{l} (E,\gamma)\in \mathrm{Bun}_T^W(\tilde{X}), \\ c_\alpha : E \overset{T,\alpha}{\times} \mathbb{G}_m|_{\tilde{X}^\alpha} \xrightarrow{\sim} \mathbb{G}_m \\ \text{for } \alpha\in\Phi, \text{ compatible w/ } \gamma \end{array} \right\}$$

$$\begin{tikzcd}
 & \mathcal{P} \arrow[d, "\text{forget } c"] \\
\mathrm{Bun}_T(\tilde{X}) \arrow[ur, dashed, "\exists ?\ \mathrm{Nm}^{\mathcal{P}}"] \arrow[r, "\mathrm{Nm}"] & \mathrm{Bun}_T^W(\tilde{X}) \\
E \arrow[r, mapsto] & \left(\bigotimes_{w\in W} w(E),\ \gamma_{can}\right)
\end{tikzcd}$$

<u>Lemma 2.</u> $\exists$ natural lift $\mathrm{Nm}^{\mathcal{P}}$.

Proof.

$$\bigotimes_{w\in W} \left(w(E) \overset{T,\alpha}{\times} \mathbb{G}_m\right)\Big|_{\tilde{X}^\alpha} \simeq \bigotimes_{w\in \langle s_\alpha\rangle\backslash W} \underbrace{\left(\left(w(E)\otimes s_\alpha w(E)\right) \overset{T,\alpha}{\times} \mathbb{G}_m\right)\Big|_{\tilde{X}^\alpha}}_{\substack{= \mathbb{G}_{m,\tilde{X}^\alpha} \\ \text{canonically}}}$$

$\square$

Put $\Lambda := X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$

and define AJ by

$$(x,\lambda) \longmapsto \mathcal{O}_{\tilde X}(\lambda x) := \mathcal{O}_{\tilde X}(x) \overset{\mathbb{G}_m,\lambda}{\times} T$$

$$\begin{tikzcd}
\tilde X \times \Lambda \arrow[r] \arrow[dr, "AJ"'] & \mathrm{Bun}_T(\tilde X) \arrow[d, "\mathrm{Nm}^{\mathcal{P}}"] \\
 & \mathcal{P}
\end{tikzcd}$$

"Abel-Jacobi map"

<u>Rem. 3</u> For $\alpha \in \Phi$, $x \in \tilde X^{\alpha}$ one has $AJ(x, \check\alpha) = e$

since $$\bigotimes_{w\in W} w(\mathcal{O}(\check\alpha x)) \simeq \bigotimes_{w \in W/\langle s_\alpha\rangle} w\big(\underbrace{\mathcal{O}(\check\alpha x) \otimes \mathcal{O}(s_\alpha(\check\alpha)x)}_{\text{trivial.}}\big)$$

<u>Lemma 4</u> AJ is multiplicative & W-equivariant, ie we have natural trafos μ and γ as follows:

$$\begin{tikzcd}
\tilde X \times \Lambda \times \Lambda \arrow[r, "{(AJ, AJ)}"] \arrow[d] & \mathcal{P} \times \mathcal{P} \arrow[d] \arrow[dl, phantom, "\swarrow\mu"] \\
\tilde X \times \Lambda \arrow[r, "AJ"'] & \mathcal{P}
\end{tikzcd}
\qquad\Bigg|\qquad
\begin{tikzcd}
\tilde X \times \Lambda \times W \arrow[r, "pr_{12}"] \arrow[d, "act"'] & \tilde X \times \Lambda \arrow[d, "AJ"] \arrow[dl, phantom, "\swarrow\gamma"] \\
\tilde X \times \Lambda \arrow[r, "AJ"'] & \mathcal{P}
\end{tikzcd}$$

$\square$

2. Definition of $\mathcal{D} : \mathcal{P}^\vee \longrightarrow \check{\mathcal{P}}$

Prop. 5 $\exists$ natural morphism $\mathcal{P}^\vee \xrightarrow{\mathcal{D}} \check{\mathcal{P}}$

over $\mathcal{B}^\circ \xrightarrow{\sim} \check{\mathcal{B}}^\circ$.

Proof.

- Recall

$$\mathcal{P}^\vee := \mathcal{H}om_{ps}(\mathcal{P}, B\mathbb{G}_m) \simeq \mathcal{P}ic^m(\mathcal{P})$$

where $\mathcal{P}ic^m(\mathcal{P})(S) := \{ (\mathcal{M}, \mu) \mid \mathcal{M} \in \mathcal{P}ic(\mathcal{P})(S),$
$\mu : \mathcal{M} \boxtimes \mathcal{M} \xrightarrow{\sim} a^* \mathcal{M} \}$
w/ natural compatibilities

"multiplicative line bdles"

- Similarly, put

$$\mathcal{P}ic^{m,W}(\tilde{X} \times \Lambda)(S) := \{ (\mathcal{M}, \gamma, \mu) \mid (\mathcal{M}, \gamma) \in \mathcal{P}ic^W(\tilde{X} \times \Lambda)(S), \; \mu : \mathcal{M} \boxtimes \mathcal{M} \xrightarrow{\sim} a^* \mathcal{M} \text{ compatible with } \gamma \}$$

- Lemma 4 gives:

$$\mathrm{AJ}^* : \mathcal{P}^\vee \longrightarrow \mathcal{P}ic^{m,W}(\tilde{X} \times \Lambda).$$

Claim:

$$
\begin{tikzcd}
\forall (\mathcal{M},\mu) \in \mathcal{P}^{\vee} \arrow[rr, dashed, "\exists \mathcal{D}"] \arrow[dd, mapsto] \arrow[d, "\mathrm{Ad}^*"] & & \check{\mathcal{P}} \arrow[d, "(\text{lemma } 1)"] \ni (E,\gamma,c) \arrow[dd, mapsto, bend left] \\
\mathcal{P}_{ic}^{m,W}(\tilde{X}\times\Lambda) \arrow[rr, "\sim"] & & \mathrm{Bun}^{W}_{\check{T}}(\tilde{X}) \\
\left((E \overset{\check{T},\lambda}{\times} \mathbb{G}_m)_{\lambda\in\Lambda}, \gamma, \mu_{can} \right) & & (E,\gamma) \arrow[ll, mapsto]
\end{tikzcd}
$$

then $\exists$ natural

Indeed:

$(\mathcal{M},\mu)$ multiplicative & $\mathrm{Ad}(x,\check{\alpha}) = e \quad \forall x \in \check{X}^{\alpha}, \ \alpha \in \Phi$ (remark 3)

$\Longrightarrow$ $\exists$ natural iso's $\quad \mathcal{M}_{\mathrm{Ad}(x,\check{\alpha})} \xrightarrow{\sim} \mathbb{G}_m$

$\Longrightarrow$ $\exists$ natural iso's

$$c_{\check{\alpha}} : E \overset{\check{T},\check{\alpha}}{\times} \mathbb{G}_m \Big|_{\check{X}^{\alpha}} \xrightarrow{\sim} \mathbb{G}_m$$

& we can take $c = (c_{\check{\alpha}})_{\check{\alpha}\in\check{\Phi}}$. $\square$

3. Reduction to coarse moduli

Basic Fact: $\mathcal{P}$ is a Beilinson 1-motive $/\mathcal{B}^0$,

more precisely:

$$\begin{array}{ccccc} W_0 & \subset & W_1 & \subset & W_2 = \mathcal{P} \\ \| & & \downarrow & & \downarrow \\ BZ(G) & & \mathrm{gr}_1^W \text{ is an abelian scheme}/\mathcal{B}^0 & & \mathrm{gr}_2^W = \pi_0(\mathcal{P}) \overset{!}{=} \Lambda/\mathbb{Z}\check{\Phi} =: \pi_1(G). \end{array}$$

Idea of proof.

coarse moduli space

Know $\ker(\mathcal{P} \longrightarrow P) = BZ(G)$,

so main point is to see $\pi_0(\mathcal{P}) = \pi_0(\mathcal{P}_b) = \pi_1(G)$ $\forall\, b \in \mathcal{B}^0(k)$.

For this use

$$\begin{tikzcd} \Lambda = \pi_0(\tilde{X}_b \times \Lambda) \arrow[r, "\pi_0(\mathrm{A}\mathcal{J})"] \arrow[d, two heads] & \pi_0(\mathcal{P}_b) \\ \pi_1(G) = \Lambda/\mathbb{Z}\check{\Phi} \arrow[ur, dashed, "\exists \varphi"'] & \end{tikzcd}$$

since $\mathrm{A}\mathcal{J}(x, \check{\alpha}) = 0$ $\forall\, x \in \tilde{X}^{\alpha}$, $\alpha \in \Phi$ (remark 3).

See Faltings, J. Alg. Geom. 2 (1993)

$\square$

Cor. 6 $\mathcal{D}$ induces iso's

$$W_0(\mathcal{P}^\vee) \xrightarrow{\sim} W_0(\check{\mathcal{P}})$$

$$\pi_0(\mathcal{P}^\vee) \xrightarrow{\sim} \pi_0(\check{\mathcal{P}}).$$

Proof.

- Duality $\Longrightarrow$ enough to discuss $\pi_0(\ldots)$.
- As abstract groups,

$$\pi_0(\mathcal{P}^\vee) \cong \mathrm{Hom}(\underbrace{\mathrm{Aut}_{\mathcal{P}}(\mathcal{E})}_{\simeq Z(G)}, \mathbb{G}_m)$$

$$\simeq \pi_1(\check{G})$$

$$\simeq \pi_0(\check{\mathcal{P}})$$

$\Longrightarrow$ enough to show $\pi_0(\mathcal{D})$ is epi, since both sides are fin. gen. abelian gps.

- Epi follows from $\qquad \check{\Lambda} = \mathrm{Hom}(T, \mathbb{G}_m)$

$$\begin{tikzcd}
 & \tilde{X} \times \check{\Lambda} \arrow[dl, dashed, "\exists \psi"'] \arrow[dr, "\check{Ad}"] & \\
\mathcal{P}^\vee \arrow[rr, "\mathcal{D}"'] & & \check{\mathcal{P}}
\end{tikzcd}$$

$\check{Ad}$ ← epi on $\pi_0(\ldots)$

with ψ constructed from universal line bdle on $(\tilde{X} \times \check{\Lambda}) \times \mathrm{Bun}_T(\tilde{X})$ $\square$